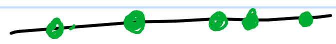
We will construct a category today as discussed last time

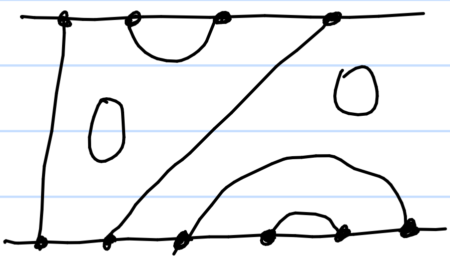
$TL^0(A)$: <u>objects</u> : finite # of points on $I = [0,1]$

x  then $|x| = $ # points

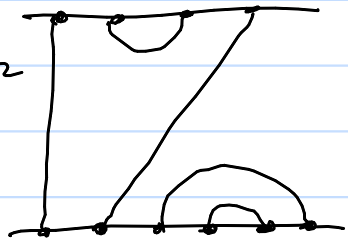
<u>morphisms</u>: $\hom(x,y) = \begin{cases} 0 & |x|+|y| \text{ is odd} \\ \cdots\cdots & \text{else} \end{cases}$

$\mathbb{C}\left[(x,y)-TL \text{ diagrams} / d\text{-isotopy}\right]$

<u>Example</u>:

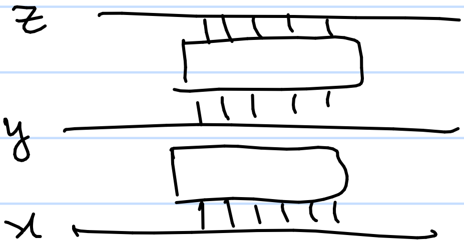


$= d^2$



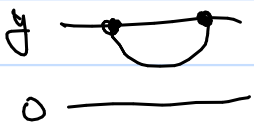
<u>Composition</u>: Stacking

read bottom up



$\mathrm{Hom}(x,y) \times \mathrm{Hom}(y,z) \longrightarrow \mathrm{Hom}(x,z)$
(extend linearly)

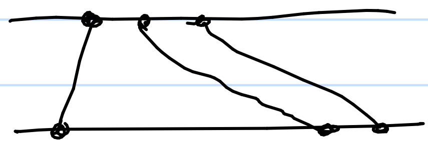
<u>Basis for morphisms</u>:
non crossing perfect matchings on $|x|+|y|/2$ points

<u>Ex</u>: 

$\in \mathrm{Hom}(0,y)$

So far this category has infinitely many objects x s.t. $|x| = n$ - fixed

But all such objects are isomorphic because if $|x| = |y|$ then we have



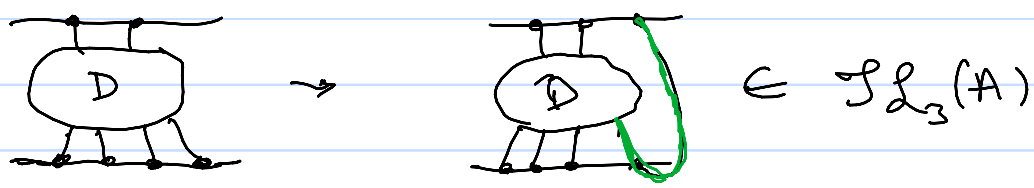
For this reason, let 1^n denote n points equally spaced.

Note:
- $\mathrm{Hom}(1^n, 1^n) = \mathrm{End}(1^n)$ is an algebra
- In fact $\mathrm{End}(1^n) \cong TL_n(A) \cong \mathcal{TL}_n(A)$
- If y has n points, then $\mathcal{TL}_n(A)$ acts on $\mathrm{Hom}(x,y)$ on right $\forall x$.

even better, the Braid group B_n acts
$\beta \in B_n \mapsto \varphi(\beta) \in TL_n$ which acts on $\mathrm{Hom}(x,y)$

- $\dim(\hom(x,y)) = \dim\left(TL_{\frac{|x|+|y|}{2}}(A)\right) < \infty$

Example:

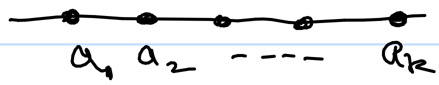


Need more, so extend
$TL^0(A)$ to $TLJ^0(A)$
s.t. $TL^0(A) < TLJ^0(A)$

$TLJ^\circ(A)$

Objects: colored points on I
(colored by $\mathbb{N} = \{0, 1, \ldots\}$

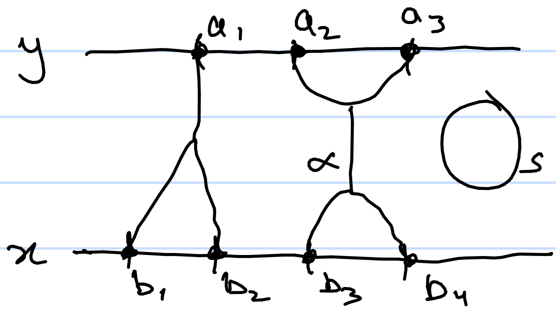
Eg: $X_{(a_1, a_2 \ldots a_k)} =$



Remark: Any $a_i = 1$ we don't color
any $a_j = 0$ we delete / ignore
(this shows how $TL^\circ(A) \leq TLJ^\circ(A)$)

Morphisms: colored uni-trivalent graphs
with conditions

uni means only one edge comes of each vertex on top & bottom line

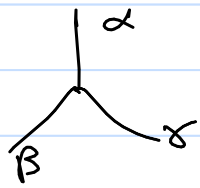


(loops are OK! but they are also colored)

In the interior is the graph is trivalent

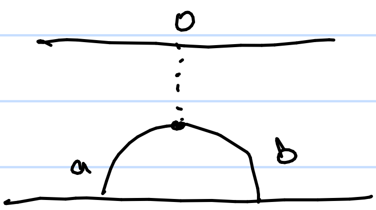
Such a graph is $\boxed{\text{admissible}}$ if

① $\alpha + \beta + \gamma \in 2\mathbb{Z}$



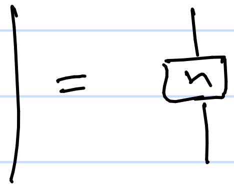
② $\alpha + \beta \geq \gamma$
$\alpha + \gamma \geq \beta$
$\gamma + \beta \geq \alpha$

Ex:



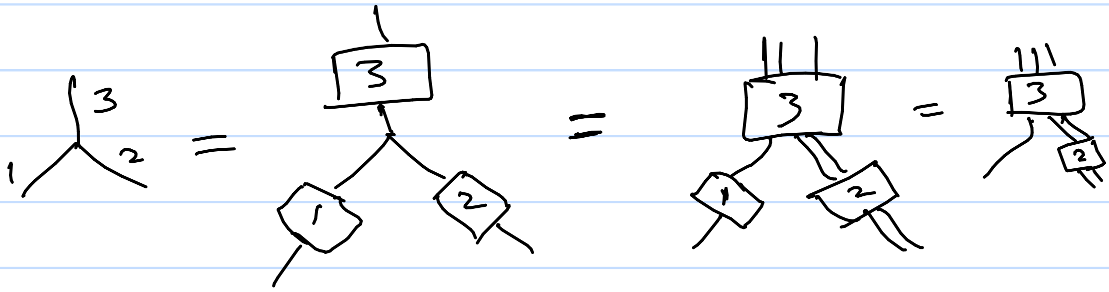
by conditions,
$a + 0 \geq b$
$b + 0 \geq a$
$\Rightarrow a = b$

Colors means JW projectors



$\in TL_n(A)$

Example:
(i)



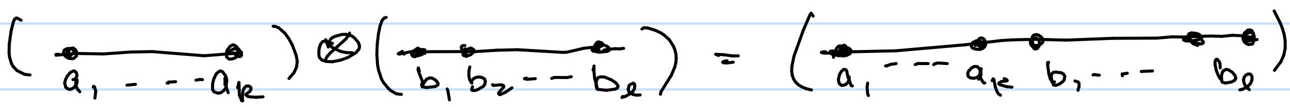
(ii)

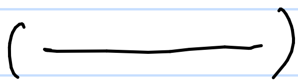
$= \Delta_s = Tr(P_s)$

- $\mathrm{Hom}\left(X_{(a_1,\dots,a_k)}, X_{(b_1,\dots,b_l)}\right)$

$$= \begin{cases} 0 & \text{if } \sum_i a_i + \sum_j b_j \text{ odd} \\ \mathbb{C}\left[\begin{array}{l}\text{admissible colored} \\ \text{uni-trivalent diagrams} \\ \text{with end points} \\ (a_1,\dots,a_k) \;\&\; (b_1,\dots,b_l)\end{array}\right] \Big/ d\text{-isotopy} & \text{else} \end{cases}$$

CLAIM: $TLJ^0(A)$ is a tensor category

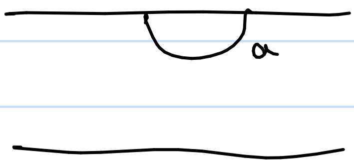


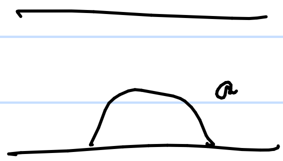
Tensor unit $=$ 

In fact, this category is rigid
$\forall$ objects, set $a^* = a$
then

we have maps

$\mathbb{1} \xrightarrow{coev} a \otimes a^*$ $\qquad\qquad$ $a^* \otimes a \xrightarrow{ev} \mathbb{1}$





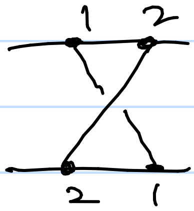
these maps satisfy snake equations

- $X_{(a_1, \dots, a_k)} = X_{a_1} \otimes X_{a_2} \otimes \dots \otimes X_{a_k}$

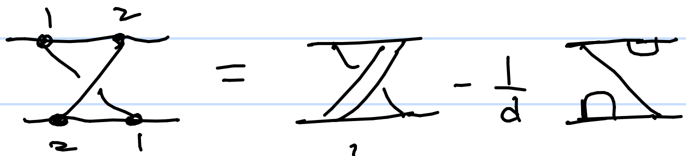
- $TLJ^{\circ}(A)$ is braided

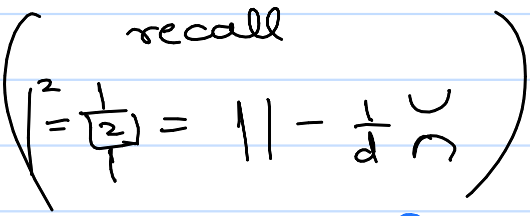
we have maps $c_{i,j} \in \mathrm{Hom}(i \otimes j, j \otimes i)$
$i, j \in \mathbb{N}$

eg: $c_{2,1}$ =

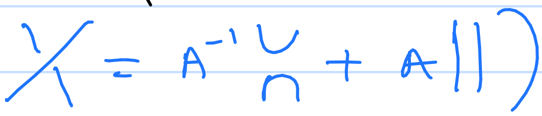


We use Kauffman bracket to interpret this

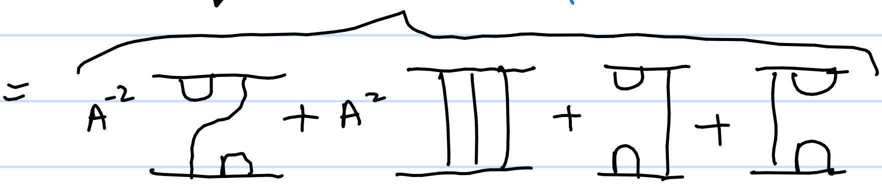




(now use the relation

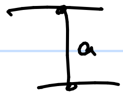


)



RIBBON:

this is ribbon element
Use Kauffman to resolve this.

- $\dim \mathrm{Hom}(a, a) = 1$ 

i.e., $a(X_a)$ is simple object in $TLJ^{\circ}(A)$

- $\dim \mathrm{Hom}(a, b) = 0$ if $a \neq b$

- 1^n is not simple
  $\dim \mathrm{Hom}(1^n, 1^n) = C_n$

- But $\dim \mathrm{Hom}(n, 1^n) = 1$

<u>Exercise:</u> $\Delta_s = (-1)^s [s+1]$
where
$$[a] = \frac{A^{2a+2} - A^{-2a-2}}{A^2 - A^{-2}}$$

Fix $r \geq 3$. Let A be a $4r$ (even) or $2r$ (odd) th root of unity (depending on parity of r)

- FACT : $Tr(P_{r-1}) = \Delta_{r-1} = 0$

- $\langle P_{r-1}, P_{r-1} \rangle = 0$ where $\langle P, Q \rangle :=$ 

- $\langle P_{r-1}, X \rangle = 0$
  ↳ any object in category

but $Tr(P_i) \neq 0$ if $1 \leq i \leq r-2$

(we get a radical that prevents from being semisimple)

<u>FACTS:</u> radical of $\langle , \rangle$ is general by P_{r-1}.

Moreover $TL_n(r) := TL_n(A) \big|_{A = 4r/2r \text{ th root of unity}}$

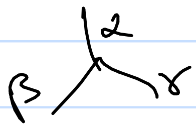
Then $TL_n(q)/\langle P_{r-1}\rangle$ is s.s. $\forall\, n$.

We now try to do the quotient with category $TLJ^0(A)$.

We're going to take a quotient category (mod out the morphism space)

(Iso classes of)
Simple objects $\longleftrightarrow$ $\mathcal{L} = \{0, 1, \dots, k\}$
where $k = r-2$

morphisms : as before but with 1 more condition

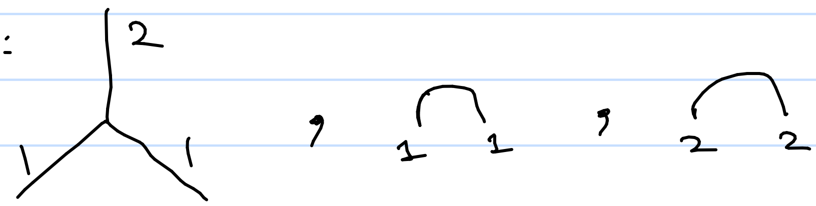


want $\boxed{\alpha+\beta+\gamma \leq 2k}$
r- admissible

eg: $r = 4 \Rightarrow k = 2$

<u>Objects</u>: $\{0, 1, 2\} = \mathcal{L}$

<u>morphisms</u>:



$\}$ $\mathrm{Hom}^r(X,Y)$

+ rotations

in this case
$A = i e^{-2\pi i/8}$

$$\bigcirc_2 = \Delta_2 = 1$$

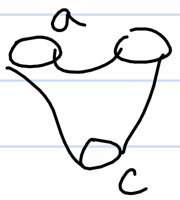
$$\bigcirc_1 = \Delta_1 = \sqrt{2}$$

We call these $TLJ^r(A)$
- spherical braided fusion
- when r is even : modular
- when r is odd

$\{0, 2, \dots, k-1\} \subseteq \mathcal{L}$
closed under $\otimes$
and is modular

• Always unitary

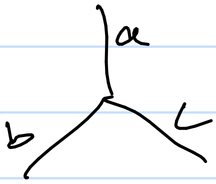
Use this category to model anyons

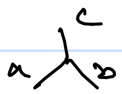
$$\mathcal{H}\Bigg($$  $$\Bigg) = \mathrm{Hom}^{r}(c, a \otimes b)$$

$a, b, c \in \mathcal{L}(r)$

<u>Nontrivial claim is</u>: with the above choice of N_{ab}^{c}, the axioms mentioned in lecture 1 are satisfied

(Reference for graphical calculus Kauffman - Lins (Princeton book))

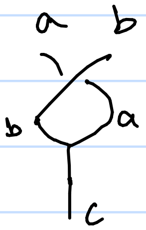
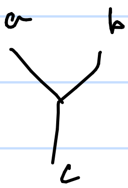


$N_{ab}^{c} = \begin{cases} 0 & \text{if } \end{cases}$  inadmissible

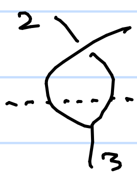
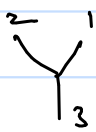
$\phantom{N_{ab}^{c} = } \;\; 1$ if admissible (r- admissible)

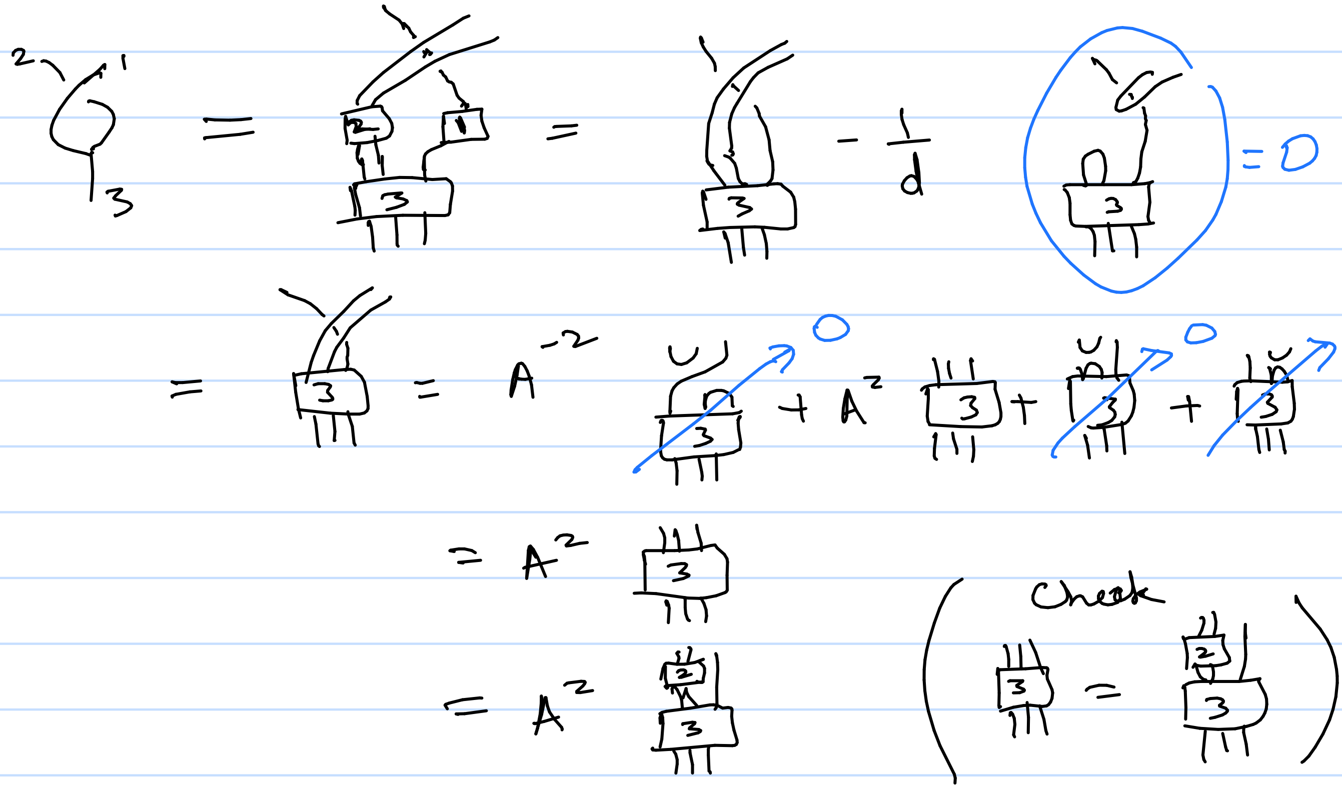
these things form a basis.

<u>BRAIDING</u>:

 $= R_{c}^{ab}$ 

since $\mathrm{Hom}^{r}(c, a \otimes b)$ is 1- dim

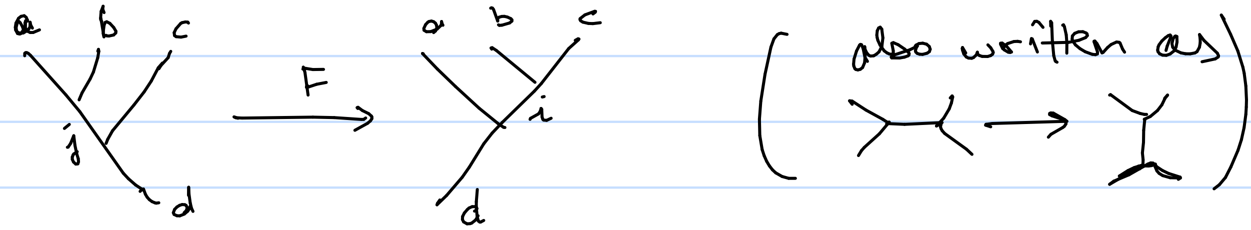
<u>Example</u>:  $= R_{3}^{21}$ 



$$\therefore \quad R^{21}_{3} = A^2$$

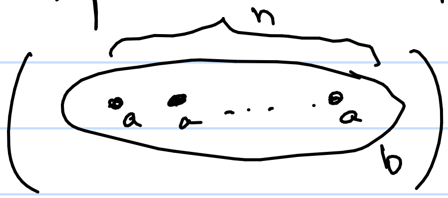
Associativity constraint
(action of Braid group)

we want to understand

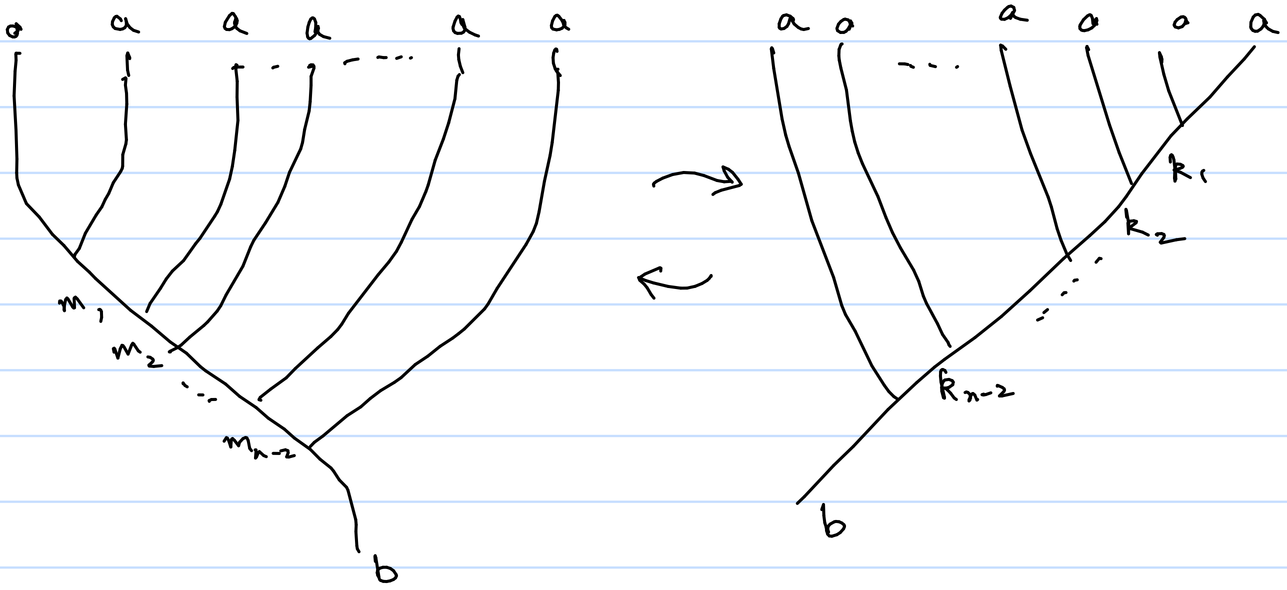


these can be worked out explicitly
(see Kauffman's book)

The space we are interested in is

$\mathcal{H}\Big($  $\Big) = \mathrm{Hom}^{*}(b, a^{\otimes n})$

Basis:



- TLJ(A) is idempotent completion of TL(A)