Problem session 1

1. Consider functors $W \otimes_A -$ and $W' \otimes_A - : \mathsf{mod}(A) \to \mathsf{mod}(B)$ induced by B-A bimodules W and W'. Show that any natural transformation between them is induced by a unique B-A bimodule map $W \to W'$.

2. An adjunction can equivalently be described by the following data.

   - functors $r : \mathsf{A} \to \mathsf{B}$ and $l : \mathsf{B} \to \mathsf{A}$
   - natural transformations $\eta : 1 \to rl$ and $\varepsilon : lr \to 1$

   such that the following diagrams commute;

$$\begin{tikzcd} & rlr \arrow[dr, "1\varepsilon"] & \\ r \arrow[ur, "\eta 1"] \arrow[rr, equal] & & r \end{tikzcd} \qquad \begin{tikzcd} & lrl \arrow[dr, "\varepsilon 1"] & \\ l \arrow[ur, "1\eta"] \arrow[rr, equal] & & l \end{tikzcd} \tag{1}$$

   The natural transformation η is called the *unit* and ε is called the *counit* of the adjunction.

3. For any adjunction $l \dashv r : \mathsf{A} \to \mathsf{B}$ with unit η and counit ε, verify bijective correspondences between natural transformations of the following kinds.

   - Between natural transformations $fr \to g$ and $f \to gl$, for any functors $f : \mathsf{B} \to \mathsf{C}$, $g : \mathsf{A} \to \mathsf{C}$ and any category C.
   - Between natural transformations $f \to rg$ and $lf \to g$, for any functors $f : \mathsf{C} \to \mathsf{B}$, $g : \mathsf{C} \to \mathsf{A}$ and any category C.

4. From any monoidal category $(\mathsf{A}, I, \otimes)$ there is a strong monoidal equivalence to the following strict monoidal category.

   The *objects* consist of a functor $t : \mathsf{A} \to \mathsf{A}$ and a natural isomorphism $\tau : (t-) \otimes - \to t(- \otimes -)$ such that for all objects X, Y, Z the following diagram commutes.

$$\begin{tikzcd} (tX \otimes Y) \otimes Z \arrow[r, "\tau_{X,Y} \otimes \mathbf{1}"] \arrow[d, "\alpha_{tX,Y,Z}"'] & t(X \otimes Y) \otimes Z \arrow[r, "\tau_{X \otimes Y, Z}"] & t((X \otimes Y) \otimes Z) \arrow[d, "t\alpha_{X,Y,Z}"] \\ tX \otimes (Y \otimes Z) \arrow[rr, "\tau_{X, Y \otimes Z}"'] & & t(X \otimes (Y \otimes Z)) \end{tikzcd} \tag{2}$$

   The *morphisms* $(t, \tau) \to (t', \tau')$ are natural transformations $\varphi : t \to t'$ such that for all objects X, Y of A the following diagram commutes.

$$\begin{tikzcd} tX \otimes Y \arrow[r, "\tau_{X,Y}"] \arrow[d, "\varphi_X \otimes \mathbf{1}"'] & t(X \otimes Y) \arrow[d, "\varphi_{X \otimes Y}"] \\ t'X \otimes Y \arrow[r, "\tau'_{X,Y}"'] & t'(X \otimes Y) \end{tikzcd} \tag{3}$$

The monoidal product of the objects (t, τ) and (t', τ') consists of the composite functor $t't$ and the natural isomorphism whose components are

$$t'tX \otimes Y \xrightarrow{\tau'_{tX,Y}} t'(tX \otimes Y) \xrightarrow{t'\tau_{X,Y}} t't(X \otimes Y)\ .$$

The monoidal product of morphisms is the Godement product of natural transformations. The monoidal unit is the identity functor with the identity natural isomorphism.

5. Show that the composite of monoidal functors is monoidal; symmetrically, the composite of opmonoidal functors is opmonoidal.

Problem session 2

1. Prove that in an adjunction $l \dashv r$ between monoidal categories, there is a bijective correspondence between the monoidal structures on r and the opmonoidal structures on l.

2. Consider an adjunction $l \dashv r$ and a strong monoidal structure (r^0, r^2) on r; then there is a corresponding opmonoidal structure (l^0, l^2) on l. Regard on r the opmonoidal structure provided by the inverses of r^0 and r^2; and on the composite functors rl and lr take the composite opmonoidal structures. Prove that with respect to these structures the unit $\eta : 1 \to rl$ and the counit $\varepsilon : lr \to 1$ of the adjunction are opmonoidal natural transformations.

3. Show that the antipode of a Hopf algebra T is an algebra homomorphism from T to the opposite algebra T^{op}. Symmetrically, show that the antipode is a coalgebra homomorphism as well from T to the opposite coalgebra.

4. Show that in any $B|B$-coring C, the image of the comultiplication is central in a suitable B-bimodule; concretely, for any $c \in C$ and $b \in B$, $c_1 \cdot (1 \otimes b) \otimes_B c_2 = c_1 \otimes_B c_2 \cdot (b \otimes 1)$.

Problem session 3

1. Show that any linear map between Frobenius algebras, which is a homomorphism of both algebras and coalgebras, is invertible.

2. Prove that the following categories are isomorphic:

   - The category whose objects are the separable Frobenius algebras over a given field; and whose morphisms are those linear maps which are both algebra and coalgebra homomorphisms.
   - The category whose objects are the separable Frobenius functors from the monoidal singleton category $\mathbb{1}$ to the monoidal category vec of vector spaces; and whose morphisms are those natural transformations which are both monoidal and opmonoidal.

3. Show that for a weak bialgebra A, the map

$$\epsilon : A \to A, \qquad a \mapsto \widehat{\epsilon}(1_{\hat{1}} a) 1_{\hat{2}} \tag{4}$$

satisfies the following identities for any $a, a' \in A$.

- $1_{\hat{1}} \otimes \epsilon(1_{\hat{2}}) = 1_{\hat{1}} \otimes 1_{\hat{2}}$ so in particular $\epsilon(1) = 1$
- $\widehat{\epsilon}(aa') = \widehat{\epsilon}(a\epsilon(a'))$ eo in particular $\widehat{\epsilon}\epsilon(a) = \widehat{\epsilon}(a)$
- $\epsilon(aa') = \epsilon(a\epsilon(a'))$ so in particular $\epsilon\epsilon(a) = \epsilon(a)$

4. Show that a monoid in the monoidal category alg of algebras over a given field k is precisely a commutative k-algebra.

5. Spell out the diagrams which the structure morphisms $\xi, \xi^0, \xi_0, \xi_0^0$ of a duoidal category $(\mathsf{A}, I, \diamond, J, \blacklozenge)$ must render commutative.