MATH 610, HOPF ALGEBRAS, FALL 2011 (2ND REVISED)
HW # 1, DUE OCTOBER 10

1. Let A be a k-space of dimension n^2 and let $C = A^*$. Let
$$\{e_{ij} \mid i, j = 1, \ldots, n\}$$
be a basis of A and let
$$\{X_{ij} \mid i, j = 1, \ldots, n\}$$
be a basis of C dual to the e_{kl}, that is, $X_{ij}(e_{kl}) = \delta_{ik}\delta_{jl}$, all i, j, k, l.

You may assume that$A$ is an (associative) algebra with multiplication $e_{ij} \cdot e_{kl} = \delta_{jk} e_{il}$ and unit element $1_A = \sum_i e_{ii}$

(a) Check that C is a coalgebra with comultiplication and counit given by
$$\Delta_C(X_{ij}) = \sum_{k=1}^{n} X_{ik} \otimes X_{kj} \text{ and } \varepsilon_C(X_{ij}) = \delta_{ij}.$$

(b) Prove that the coalgebra structure on C is the dual of the algebra structure on A.

Thus $A = M_n(k)$, the usual algebra of $n \times n$ matrices over k, if and only if $C = A^*$ is the "matrix coalgebra" mentioned in class.

2. Let $B = \mathcal{O}(M_n(k))$ be the bialgebra of polynomial functions on $M_n(k)$. That is, $B = k[X_{ij}]$ as polynomials in the X_{ij}, where the X_{ij} act on $M_n(k)$ as in Problem 1. B becomes a coalgebra by using the maps Δ, ε on the X_{ij} as in Problem 1 and extending them multiplicatively to B. Let $\mathbb{X} = [X_{ij}] \in M_n(B)$, the "generic" $n \times n$ matrix.

Prove that $g = Det(\mathbb{X}) \in B$ is a group-like element (that is, $\varepsilon(g) = 1$ and $\Delta(g) = g \otimes g$) but that g is not invertible in B. Conclude that B does not have an antipode S. (Hint: use Problem 1 for a non-computational proof)

3. Let B be a bialgebra, and let $P(B)$ denote the set of primitive elements in B; that is,
$$P(B) = \{x \in B \mid \Delta(x) = x \otimes 1 + 1 \otimes x \text{ and } \varepsilon(x) = 0.$$

(a) Check that $P(B)$ becomes a Lie algebra by defining $[x, y] = xy - yx$, for any $x, y \in P(B)$.

(b) For $B = k^G$, G a group, show that $f \in P(B) \iff f$ is a homomorphism from G to $k, +$, the additive group of k.

(c) If B is a finite-dimensional bialgebra of characteristic 0, show that $P(B) = 0$. (This is non-trivial)

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4. Let $k = \mathbb{C}$ and let $H = H_4$ be Sweedler's 4-dim Hopf algebra. That is,

$$H = {}_k\langle g, x \mid g^2 = 1, x^2 = 0, xg = -gx\rangle \tag{1}$$

where $\Delta(g) = g \otimes g$, $\Delta(x) = x \otimes 1 + g \otimes x$, $\varepsilon(g) = 1$, $\varepsilon(x) = 0$, $S(g) = g^{-1}$, and $S(x) = -g^{-1}x$.

We claim that $H^* \cong H$ as Hopf algebras ; that is, we claim we may write

$$H^* = {}_k\langle G, X \mid G^2 = \varepsilon, X^2 = 0, XG = -GX\rangle \tag{2}$$

where $\Delta(G) = G \otimes G$, $\Delta(X) = X \otimes \varepsilon + G \otimes X$, $\varepsilon_{H^*}(G) = \langle G, 1\rangle = 1$, and $\varepsilon_{H^*}(X) = \langle X, 1\rangle = 0$, $S(G) = G$, and $S(X) = -GX$.

To prove that $H^* \cong H$, *define* G, $X \in H^*$ on a basis of H by

$$\langle G, 1\rangle = 1,\ \langle G, g\rangle = -1,\ \langle G, x\rangle = 0,\ \langle G, gx\rangle = 0, \tag{3}$$

$$\langle X, 1\rangle = \langle X, g\rangle = 0,\ \langle X, x\rangle = 1,\ \langle X, gx\rangle = -1. \tag{4}$$

Using (3) and (4), show:

(a) $XG = -GX$ in H^*, where the product in H^* is convolution. Note it is enough to check this on a basis of H.

(b) $\Delta(G) = G \otimes G$ and $\Delta(X) = X \otimes \varepsilon + G \otimes X$. For this part, check their action on a basis of $H \otimes H$.

5. Summation notation practice:

In a coalgebra C, the summation notation version of the properties of the counit may be stated as

$$c = \sum_{(c)} \varepsilon(c_1)c_2 = \sum_{(c)} \varepsilon(c_2)c_1.$$

for any $c \in C$. Using this and coassociativity, show that for any $c \in C$,

(a) $c = \sum_{(c)} \varepsilon(c_1) \otimes \varepsilon(c_2) \otimes c_3 = \sum_{(c)} \varepsilon(c_1)\varepsilon(c_2)c_3$.

(one assumes that the tensor products of scalars is just the product)

In a Hopf algebra H, the fact that the antipode S is the $*$-inverse of the identity map id on H says that for all $h \in H$,

$$\sum_{(h)} h_1 S(h_2) = \sum_{(h)} S(h_1)h_2 = \varepsilon(h)1_H.$$

Using this, as well as other known properties of S, Δ, and ε, show that

(b) $\sum_{(h)} h_1 \otimes (S(h_2))h_3 = h \otimes 1$, for any $h \in H$;

(c) $\sum_{(h)} (1 \otimes S(h_3)h_1)\Delta S(h_2) = (S \otimes S)\Delta(h)$, for any $h \in H$.