MATH 610, HOPF ALGEBRAS, FALL 2011
HW #2, DUE NOVEMBER 14

1. (note: computations awful) Consider the n^2-dimensional Taft algebra $H = T_{n^2,\omega}$, for ω a fixed primitive n^{th} root of 1. That is,

$$H = k\langle gx|\ g^n = 1, x^n = 0, xg = \omega gx\rangle \tag{1}$$

where $\Delta(g) = g \otimes g$, $\Delta(x) = x \otimes 1 + g \otimes x$, $\varepsilon(g) = 1$, and $\varepsilon(x) = 0$. Similarly to the case $n = 2$ on HW #1, $H^* \cong H$; thus we may write

$$H^* = k\langle G, X|\ G^n = \varepsilon, X^n = 0, XG = \omega GX\rangle \tag{2}$$

where $\Delta(G) = G \otimes G$, $\Delta(X) = X \otimes \varepsilon + G \otimes X$, $\varepsilon_{H^*}(G) = \langle G, 1\rangle = 1$, and $\varepsilon_{H^*}(X) = \langle X, 1\rangle = 0$. The dual pairing between H and H^* is determined by

$$\langle G, g\rangle = \omega^{-1},\ \langle G, x\rangle = 0,\ \langle X, g\rangle = 0,\ \text{and } \langle X, x\rangle = 1. \tag{3}$$

Note that $\Gamma = x^{n-1}(\sum_{i=0}^{n-1} g^i)$ is a right integral of H and that $\lambda = (\sum_{i=0}^{n-1} G^i)X^{n-1}$ is a left integral of H^*.

Find all the "Frobenius data" for H: that is, give a Frobenius system $\{f, r_i, l_i\}$ for H using $f = \lambda$, find the modular element $\beta \in H^*$, and then find the Nakayama automorphism η for H with respect to this system (recall that β is determined by $h\Gamma = \beta(h)\Gamma$, for all $h \in H$).

2. Let G be a finite group and let $D(G) = k^G \bowtie kG$ be the Drinfel'd double of G. That is, as a coalgebra, $D(G) = k^G \otimes kG$ is the tensor product of k^G and kG. As an algebra, $D(G)$ is the semi-direct product $k^G \# kG$, where G acts on k^G by $g \cdot p_x = p_{gxg^{-1}}$.

(a) We know that $t = \sum_{g\in G} g$ is an integral in kG and p_1 is an integral in k^G. Show that $p_1 \bowtie t$ is both a left and right integral in $D(G)$, and so $D(G)$ is unimodular.

(b) Note that since $D(G)$ is a tensor product of algebras, $D(G)^*$ is the tensor product of the algebras $(k^G)^* = kG$ and $(kG)^* = k^G$. Find an integral in $D(G)^*$.

(c) Find a set of "Frobenius data" for $D(G)$: that is, find a Frobenius system $\{f, r_i, l_i\}$ and the corresponding Nakayama automorphism ν of $D(G)$.

Continued $\longrightarrow$

3. A non-Frobenius algebra:

Let p be a prime > 2 and let $G = C_p \times C_p$, where C_p is the cyclic group of order p. Let k be a field of characteristic p, and let $H = kG$, the group algebra.

(a) Show that the radical of H is $N = H^+ = Ker(\varepsilon)$ (and so N has codimension 1). Since $1 - g \in H^+$ for all $g \in G$, it then follows that $N^2 \neq 0$.

(b) Show that the algebra $A = H/N^2$ is NOT a Frobenius algebra (hint: if it were Frobenius, then $A \cong A^*$ as left (and right) A-modules).

4. We consider the quantum version of $B = \mathcal{O}(M_n(k))$ from HW #1, when $n = 2$. For a parameter q, $\mathcal{O}_q(M_2(k))$, the *coordinate ring of quantum* $M_2(k)$, is defined as follows:

$$\mathcal{O}_q(M_2(k)) := \langle a, b, c, d \rangle$$

subject to the relations

$$ba = q^{-2}ab \qquad ca = q^{-2}ac \qquad bc = cb$$

$$db = q^{-2}bd \qquad dc = q^{-2}cd \qquad ad - da = (q^2 - q^{-2})bc$$

Define a comultiplication and co-unit on $\mathcal{O}_q(M_2(k))$ by

$$\Delta(a) = a \otimes a + b \otimes c, \quad \Delta(b) = a \otimes b + b \otimes d,$$
$$\Delta(c) = c \otimes a + d \otimes c, \quad \Delta(d) = c \otimes b + d \otimes d$$
$$\varepsilon(a) = \varepsilon(d) = 1, \quad \varepsilon(b) = \varepsilon(c) = 0.$$

Then $\mathcal{O}_q(M_2(k))$ becomes a bialgebra by extending Δ and ε to $\mathcal{O}_q(M_2(k))$ multiplicatively. If we write

$$\tilde{X} = \begin{bmatrix} a & b \\ c & d \end{bmatrix} = \begin{bmatrix} \tilde{X}_{11} & \tilde{X}_{12} \\ \tilde{X}_{21} & \tilde{X}_{22} \end{bmatrix},$$

the coproduct and co-unit are given by $\Delta(\tilde{X}_{ij}) = \sum_k \tilde{X}_{ik} \otimes \tilde{X}_{kj}$, and $\varepsilon(\tilde{X}_{ij}) = \delta_{ij}$, as for $\mathcal{O}(M_2(k))$.

(a) The map $\Phi : \mathcal{O}(M_2(k)) \to \mathcal{O}_q(M_2(k))$ given by $\Phi(X_{ij}) = \tilde{X}_{ij}$ on generators and extending multiplicatively on an ordered basis is a k-space isomorphism, since by Kassell p 81, $\{a^i b^j c^s d^t\}$ is a basis for $\mathcal{O}_q(M_2(k))$. Show that it is not an isomorphism of coalgebras.

(b) Now define $\det_q X := ad - q^2 bc$. Check that $\det_q \tilde{X}$ is a group-like element which is not invertible (unlike the case $q = 1$, I do not know an elegant argument for this). Thus as for $\mathcal{O}(M_2(k))$, $\mathcal{O}_q(M_2(k))$ is not a Hopf algebra.