MATH 610, HOPF ALGEBRAS, FALL 2011
HW #3, DUE DECEMBER 7

Throughout, $k = \mathbb{C}$.

1. Let G be a finite abelian group. Let $\hat{G}$ be the set of 1-dimensional characters on G; that is, $\chi : G \to k$ is in $\hat{G} \iff \chi(gh) = \chi(hg)$, for all $g, h \in G$.

(a) If G is cyclic, prove that $G \cong \hat{G}$ as groups. (hint: say $G = \langle a \rangle$ with $0(a) = n$, and let $\omega \in \mathbb{C}$ be a primitive n^{th} root of 1. Define $\Phi : G \to \mathbb{C}$ by $a^i \mapsto \omega^i$, and show that $G \to \hat{G}$, determined by $a \mapsto \Phi$, is a group isomorphism).

(b) If G, H are finite abelian, show $\widehat{G \times H} \cong \hat{G} \times \hat{H}$. This is a special case of Theorem 10 in Serre's book; you may extract the relevant parts.

(c) Conclude from (a), (b), and the Basis Theorem for Abelian Groups that $G \cong \hat{G}$ for any finite abelian group G.

(d) Conclude that for G finite abelian, $\mathbb{C}G \cong \mathbb{C}^G$ as Hopf algebras.

2. Let $H_4 = T_{4,-1}$, as in #4 on HW #1. . Define

$$R = \frac{1}{2}\big(1 \otimes 1 + 1 \otimes g + g \otimes 1 - g \otimes g\big) \in H \otimes H.$$

(a) Show that $\tau(R) = R = R^{-1}$.

(b) Show that (H_4, R) is quasi-triangular (see definition in [M, 10.1.5] or [K, p 173]). Note that to check almost-cocommutative (this is called quasi-cocommutative in [K]), it suffices to check on the generators g and x.

(c) For any quasi-triangular Hopf algebra (H, R) with $R = \sum_i a_i \otimes b_i$, recall from class (and [M, 10.1.4]) that if $u := \sum_i S(b_i)a_i$ (the "Drinfel'd element") then a theorem of Drinfel'd says that S^2 is an inner automorphism on H via u; that is, $S^2(h) = uhu^{-1}$ for all $h \in H$. Verify Drinfeld's theorem for H_4.