CIMPA BOGOTÁ: HOPF ALGEBRAS MINI-COURSE

Homework #1

Let k be a field. Let H be a finite-dimensional Hopf algebra over k with:

- multiplication $m : H \otimes H \to H$,
- unit $u : k \to H$,
- comulitplication $\Delta : H \to H \otimes H$ with Heinemann-Sweedler notation

$$\Delta(h) = \sum h_{(1)} \otimes h_{(2)} \quad (h \in H)$$

- counit $\epsilon : H \to k$, and
- antipode $S : H \to H$.

(1) (a) Show that S is the inverse of id_H in the convolution algebra $\mathrm{Hom}(H, H)$.
(b) Show that $m^{op} \circ (S \otimes S)$ and $S \circ m$ are right and left inverses, respectively, of m in the convolution algebra $\mathrm{Hom}(H \otimes H, H)$.

(2) $G(H^*) = \{\alpha | \Delta_{H^*}(\alpha) = \alpha \otimes \alpha\}$ is the group of group-like elements of the dual H^*. These satisfy $\alpha(ab) = \alpha(a)\alpha(b)$, hence an algebra homomorphisms $H \to k$. Show that $G(H^*)$ is isomorphic to the group of algebra homomorphisms $H \to k$.

(3) Let $\mathrm{Rep}(H)$ be the category of the finite-dimensional representations over k, i.e. left H-modules.
(a) For $V, W \in \mathrm{Rep}(H)$, show that $V \otimes W \in \mathrm{Rep}(H)$ via the action given by Δ:

$$h \cdot (v \otimes w) = \sum h_{(1)}v \otimes h_{(2)}w$$

(b) For $U, V, W \in \mathrm{Rep}(H)$, show that the vector space associativity isomorphism $(U \otimes V) \otimes W \to U \otimes (V \otimes W)$ is a map in $\mathrm{Rep}(H)$, i.e. is an H-module map. (Hint: this follows from coassociativity of Δ.)
(c) Let $\mathbf{1} = k$ be the H-module with the action given by ϵ:

$$h \cdot 1_k = \epsilon(h) \quad (h \in H)$$

Show that $\mathbf{1} \otimes V \cong V \cong V \otimes \mathbf{1}$ are isomorphisms in $\mathrm{Rep}(H)$.

(4) Show that, for any vector space V, the space $V \otimes H$ is a right Hopf module over H. (The H-action and H-coaction are defined from the multiplication and comultiplication of H, i.e. $\mathrm{id}_V \otimes \Delta : V \otimes H \to V \otimes H \otimes H$.)

(5) Show that H^* is a right Hopf module over H via the following action:

$$f \leftharpoondown a := S(a) \rightharpoonup f$$
$$(f \leftharpoondown a)(b) = f(bS(a))$$

and coaction:

$$\rho : H^* \to H^* \otimes H$$
$$f \mapsto \sum f^{(1)} \otimes f^{(2)}$$

where

$$g * f = \sum f^{(1)} g(f^{(2)}) \quad \forall g \in H^*$$

(6) Let Λ be a left integral of H. For $a \in H$ we have that Λa is still a left integral of H. Therefore $\Lambda a = \Lambda\alpha(a)$ for some $\alpha \in H^*$. Show that *alpha* is a group-like element of H^*.

Homework #2

(1) Suppose $char(k) \neq 0$. It was shown by Etingof-Gelaki that if $tr(S^2) \neq 0$ then $S^2 = \mathrm{id}_H$. Show that the converse statement is false.

(2) Consider the isomorphism of finite dimensional vector spaces $j : V \to V^{**}$ given by $v \mapsto \hat{v}$ where $\hat{v}(f) = f(v)$ for $f \in V^*$ and $v \in V$. Now suppose $V \in \mathrm{Rep}(H)$ and show that j is a morphism in $\mathrm{Rep}(H)$ provided that $S^2 = \mathrm{id}_H$.

(3) Consider the n^2-dimensional Taft algebra $H = T_{n^2,\omega}$ defined as follows. Let ω be a primitive n^{th} root of unity. Then:

$$T_{n^2,\omega} = k\langle g, x | g^n = 1, x^n = 0, xg = \omega g x\rangle$$

with $\Delta(g) = g \otimes g$, $\Delta(x) = x \otimes 1 + g \otimes x$, $\epsilon(g) = 1$, $\epsilon(x) = 0$. Show that the non-zero left and right integrals of $T_{n^2,\omega}$ are linearly independent.

(4) Recall that if (H, R) is a quasi-triangular Hopf algebra then $\mathrm{Rep}(H)$ is braided. Show the converse: if $\mathrm{Rep}(H)$ admits a braiding $c_{V,W} : V \otimes W \to W \otimes V$ then $c_{H,H}(1 \otimes 1)^{op} = R$ a universal R-matrix for H.