For this lecture, $\mathbb{k} = \mathbb{C}$

§1 Exponent of a Hopf alg. over $\mathbb{C}$

Thm: (Etingof) Let H be a f.d. Hopf alg. over $\mathbb{C}$ and $R \in H \otimes H$ a universal R-matrix, then $(R^{21} R)^N$ is unipotent for some $N \in \mathbb{N}$.
or $(R^{21} R)^N - 1$ is nilpotent.

Defn: Let H be a f.d. Hopf algebra
$\exp(H)$ = the smallest possible integer N s.t. $(R^{21} R)^N$ is unipotent where R is the universal R-matrix of $D(H)$. $R = a_i \otimes b_i$

Thm [Etingof - Gelaki] H f.d. Hopf algebra over $\mathbb{C}$.

1) Let $u \in D(H)$ be the Drinfeld element $u = \sum S(b_i) a_i$
   Then $\exp(H)$ = smallest ~~positive~~ integer n s.t. u^N is unipotent.

2) $\exp(H)$ is an invariant of the tensor category $\mathrm{Rep}(H)$.
   i.e. $\mathrm{Rep}(H) \xrightarrow[\text{tensor}]{\simeq} \mathrm{Rep}(K)$, then $\exp(H) = \exp(H)$

3) $\mathrm{ord}(S_H^2) \mid \exp(H)$

4) If H is pointed, then $\exp(H) = \exp(G(H))$

Question: Is $\mathrm{ord}(S_H^2)$ an invariant of $\mathrm{Rep}(H)$?
(still open)

<u>Remark</u>: If $H = \mathbb{C}[G]$, then $\exp(H) = \exp(G)$

what is this?

§2 Exponents of semisimple Hopf algebras

<u>Ex</u>: If H is s.s., then $D(H)$ is a s.s. Hopf algebra.

If u is the Drinfeld element of $D(H)$

$S^2(h) = u h u^{-1}$

But since, H is semisimple, $S^2_{D(H)} = Id$

$\Rightarrow u \in \text{Center}(D(H))$

If $W \in \text{Rep}(D(H))$ irreducible, then $u_W = \underbrace{u_W}_{\text{this is a scalar}} \cdot Id_W$

(*) If H is s.s., then $\text{Rep}(D(H))$ is a <u>modular</u> tensor category.
- follows from Takeuchi, Müger
(for Hopf alg) (for Tensor cats.)

θ is a ribbon structure

$id_e \longrightarrow id_e$

$u \in D(H)$ defines the ribbon structure

$u_V : V \rightarrow V$

Thm [Vafa] If $\mathcal{C}$ is a modular tensor cat, then the ribbon structure of $\mathcal{C}$ has finite order

$\Rightarrow u^N = Id$ for some $N \in \mathbb{N}$.

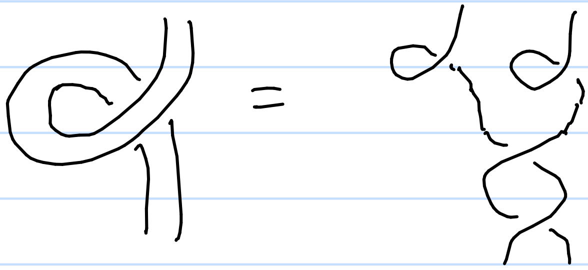
In this case (H s.s.)

$$\exp(H) = \operatorname{ord}(u)$$

$\Rightarrow R^{21}R$ has finite order and $\operatorname{ord}(R^{21}R) = \operatorname{ord}(u)$

this follows from ribbon equation

$$\Delta u = (u \otimes u)(R^{21}R)$$



(Think of all this in terms of finite groups)

Open question: $\exp(H) \mid \dim(H)$

Thm: [EG] $\exp(H) \mid \dim(H)^3$

§ Exponent via Frobenius - Schur indicator

Recall the indicators of finite group

If $V \in Rep(G)$, $\nu_n(V) = \frac{1}{|G|} \sum_{g \in G} \chi_V(g^n)$

If $n = \exp(G)$

$$\nu_n(V) = \frac{1}{|G|} \sum_{g \in G} \chi_V(1) = \chi_V(1) = \dim(V)$$

These FS indicators are periodic

Q If $\nu_n(V) = \dim(V)$ for all $V \in Rep(G)$ is n a multiple of $\exp(G)$?

Ans: Yes (exercise)

$\therefore \exp(G) =$ smallest positive integer n s.t. $\nu_n(V) = \dim V \ \forall V \in Rep(G)$

(so we've obtained a categorical defn. of exponent of G)

The above Q is true more generally.

Q Is $\underset{s.s.}{\exp(H)} =$ smallest positive integer N s.t. $\nu_N(V) = \dim V$ for all $V \in Rep(H)$. ?

Ans: Yes. Infact it is true for any spherical fusion category.

§4 Another formula of indicator

$H \hookrightarrow D(H)$ Hopf algebra

If $V \in Rep(H)$, then

$Ind(V) = D(H) \otimes_H V$

for any $V \in Rep(H)$

(we need H s.s. only for D(H) to be s.s.)

Thm [Kashima - Sommerhauser - Zhu]

Let H be a s.s. Hopf algebra over $\mathbb{C}$ & $V \in Rep(H)$. Then

$$\nu_n(V) = \frac{1}{\dim(H)} \chi_{Ind(V)}(u^n) \in \mathbb{Z}[\zeta_{\exp(H)}]$$

where u is the Drinfeld element of $D(H)$.

Answer to question about $\exp(H)$:

If $N = \exp(H)$, $u^N = \mathrm{Id}$

then $$\nu_N(V) = \frac{1}{\dim H} \chi_{\mathrm{Ind}(V)}(u^n)$$
$$= \frac{1}{\dim H} \chi_{\mathrm{Ind}(V)}(1)$$

$$\left(\begin{aligned} &=: \dim(\mathrm{Ind}(V)) \\ &= \dim(H^*) \dim(V) \\ &= \dim(H) \dim(V) \end{aligned}\right) = \frac{1}{\dim(H)} \dim(\mathrm{Ind}(V)) = \dim(V)$$

- If $\nu_n(V) = \dim V$ for all $V \in \mathrm{Rep}(H)$

use $V = H$, then $\mathrm{Ind}(V) = D(H)$

$$\Rightarrow \dim H = \sum_{W \in \mathrm{Irr}(D(H))} (\dim W)(\dim W)\, u_W^n$$

$$\Rightarrow (\dim H)^2 = \sum_W (\dim W)^2 u_W^n$$

But $$(\dim H)^2 = \sum_{W \in \mathrm{Irr}(D(H))} (\dim W)^2$$

$$\Rightarrow u_W^n = 1$$

$$\Rightarrow u_{D(H)}^n = 1$$

$\Rightarrow n$ is multiple of $\exp(H)$

$$\nu_n(V) = \chi_V(\Lambda^{[n]})$$
where Λ is the normalized integral.

<u>Note:</u> ribbon structure has same order as $R^{21}R$ for Hopf algebra.

Not true for even quasi-Hopf algebra.

$\{\nu_n(V) \mid n \in \mathbb{N}\}$

period of this sequence.

§5 Class equation of Hopf algebra

G - finite group

$$\sum_a |K(a)| = |G|$$

we want to generalize it to Hopf algebras.

Let $V \in \operatorname{Rep}(H)$ be irreducible (s.s.)

$$\nu_1(V) = \chi_V(\Lambda^{[1]}) = \chi_V(\Lambda)$$

where Λ is the normalized integral of H

$\varepsilon(\Lambda) = 1$ $\quad (\Lambda \cdot \Lambda = \varepsilon(\Lambda)\Lambda = \Lambda)$

Λ is idempotent

$$\therefore \chi_V(\Lambda) = \begin{cases} 0 & \text{if } V \not\cong \mathbb{1} \\ 1 & \text{if } V \cong \mathbb{1} \end{cases}$$

trivial rep, it corresponds to Λ

$$\Rightarrow \nu_1(V) = \chi_V(\Lambda) = \delta_{\mathbb{1},V}$$

Also, $\nu_1(V) = \frac{1}{\dim H} \chi_{\operatorname{Ind}(V)}(u^1)$

If $V \not\cong \mathbb{1}$, $\quad 0 = \frac{1}{\dim H} \chi_{\operatorname{Ind}(V)}(u)$

or $\quad 0 = \chi_{\operatorname{Ind}(V)}(u)$

$$\operatorname{Hom}_{D(H)}(\operatorname{Ind}(V), W) \cong \operatorname{Hom}_H(V, \operatorname{Res}^{D(H)}_H W)$$

But $\operatorname{Ind}(V) = \bigoplus_{W \in \operatorname{Irr}(D(H))} [\operatorname{Ind}(V), W]_{D(H)} W$

$$= \bigoplus_W [W : V]_H W$$

$\therefore \quad \mathcal{D} = \sum_W [W:V]_H \dim W \, u_W$

When $V \cong \mathbb{1}$, $\nu_1(\mathbb{1}) = 1$

$$= \frac{1}{\dim H} \sum_{W \in \mathrm{Irr}(D(H))} [W:\mathbb{1}]_H \dim W \, u_W$$

$$\Rightarrow \dim H = \sum_W [W:\mathbb{1}]_H (\dim W) \, u_W \quad \text{——— (1)}$$

$$\dim(\mathrm{Ind}(\mathbb{1})) = \dim(H)$$

using induction restriction $\quad \| \quad \sum_W [W:\mathbb{1}] \dim W$

Hence, $\dim(H) = \sum_W [W:\mathbb{1}] \dim W$ ——— (2)

(1) + (2) $\Rightarrow u_W = 1$

or $u_{\mathrm{Ind}(\mathbb{1})} = id_{\mathrm{Ind}(\mathbb{1})}$ (ribbon structure)

CLASS EQUATION $\quad \dim(H) = \sum_W [W:\mathbb{1}] \dim W$

Why class equation?

$H = \mathbb{C}[G]$

$D(H) = \mathbb{C}[G]^* \otimes \mathbb{C}[G]$ as vector space

$\{g \in G\}$ basis for $\mathbb{C}[G]$

$\{e(g) \mid g \in G\}$ dual basis for $\mathbb{C}[G]^*$

- $(e(g) \otimes x) \cdot (e(h) \otimes y) = \delta_{g^x, h} \, e(g) \otimes xy$

- $\Delta(e(g) \otimes x) = \sum_{ab=g} (e(a) \otimes x) \otimes (e(b) \otimes g)$

Irr reps of $D(G)$
indexed by $(K(a), V)$ where V is an irr. of $C_G(a)$

Define $M(a) = \{ e(a^{x^{-1}}) \otimes x \mid x \in G \}$
is a left ideal of $D(G)$ which admits right action of $C_G(a)$

$M(a) \in D(H) - C_G(a)$ bimodule

$M(a) \otimes_{\mathbb{C}[C_G(a)]} V$ is an irreducible of $\mathrm{Rep}(D(G))$

claim: $[(K(a), V) : \mathbb{1}]_{\mathbb{C}G} \neq 0$
iff $V = \mathbb{1}$ or

$$(K(a), \mathbb{1}) \longleftrightarrow \mathrm{Ind}_{C_G(a)}^{G} \mathbb{1}$$

Then $[(K(a), \mathbb{1}), \mathbb{1}]_{\mathbb{C}G} = |K(a)|$

as a consequence, we recover
$$|G| = \sum_a |K(a)|$$

§6 Rep(D(H)) is a modular tensor category

Let $\{\chi_1, \dots, \chi_n\}$ be the irreducible characters of $D(H)$.

$$S_{ij} = (\chi_i^* \otimes \chi_j)(R^{21}R)$$

$S = [S_{ij}]$ is invertible

$T = [u_i \delta_{ij}]$ where $u_i = \dfrac{\chi_i(u)}{\chi_i(1)}$

$$s = \frac{1}{\dim H} S$$

$$t = T$$

We get a map

$$SL(2,\mathbb{Z}) \longrightarrow GL(n, \mathbb{C})$$

$$\begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix} \longmapsto s$$

$$\begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix} \longmapsto t$$

is an ordinary representation

$$\begin{aligned} S_{i1} &= (\chi_1^* \otimes \chi_i)(R^{21}R) \\ &= (\varepsilon \otimes \chi_i)(R^{21}R) \\ &= \chi_i(1) = \dim(V_i) \end{aligned}$$

Using the S-matrix of $D(H)$, one can show that $\dfrac{\dim(D(H))}{(\dim V_i)^2} \in \mathbb{Z}$

(first proof by Etingof – Gelaki)

$$\Rightarrow \quad \frac{(\dim H)^2}{(\dim V_i)^2} \in \mathbb{Z}$$

$$\Rightarrow \quad \dim V_i \mid \dim H$$

where $V_i \in \mathrm{Irr}(D(H))$

Open question (Kaplansky)
If $V \in \mathrm{Rep}(H)$ is irreducible, then $\dim(V) \mid \dim H$

Class equation of Hopf algebra

$$\dim H = \sum_{W \in \mathrm{Irr}(D(H))} \dim(W)\,[W:\mathbb{1}]$$

Note that $\dim(W) \mid \dim(H)$

In particular if $\dim(H) = p^n$
$\dim(W)$ is a p-power

(among the W, trivial object must be there
it means .... figure out)