§ 1 Grouplike elements of $D(H)$ or $D(H)^*$

$$Z(\mathrm{Rep}(H)) \underset{\substack{\text{braided}\\ \text{monoidal}\\ \text{equivalence}}}{\cong} \mathrm{Rep}(D(H))$$

$$\underbrace{G(D(H)^*)}_{\text{algebraic}} = \text{1-dim representation of } D(H)$$
$$= \text{invertible objects of } \underbrace{Z(\mathrm{Rep}(H))}_{\text{categorical}}$$

Let H be a f.d. Hopf alg over $\mathbb{k}$

$D(H) = H^* \otimes H$ as vector space

- $(f \otimes a)(g \otimes b) = f * g(S^{-1}(a_{(3)}) ? a_{(1)}) \otimes a_{(2)} b$

$$= f * (a_{(1)} \rightharpoonup g \leftharpoonup S^{-1}(a_{(3)})) \otimes a_{(2)} b$$

- $\Delta(f \otimes a) = (f_{(2)} \otimes a_{(1)}) \otimes (f_{(1)} \otimes a_{(2)})$

- $\varepsilon_{D(H)}(f \otimes a) = f(1)\, \varepsilon(a)$

- $$\left.\begin{array}{l} \tau_1 : H \hookrightarrow D(H) \\ \quad a \longmapsto \varepsilon \otimes a \\ \tau_2 : (H^{op})^* \hookrightarrow D(H) \\ \quad f \longmapsto f \otimes 1 \end{array}\right\} \text{Hopf algebra embeddings}$$

where $H^{op} = (H, m_H^{op}, 1_H, \Delta, \varepsilon, S_H^{-1})$

$D(H)^* \equiv H \otimes H^*$ as vector space

$$\langle a \otimes f, \underset{H^* \otimes H}{g \otimes b} \rangle = g(a) \otimes f(b)$$

Thm: [Radford]

(i) Every grouplike element of $D(H)^*$ is of the form $g \otimes \alpha$ where $g \in G(H)$ and $\alpha \in G(H^*)$

(ii) If $g \otimes \alpha \in G(D(H)^*)$, then $\alpha \otimes g \in G(D(H)) \cap \text{Center}(D(H))$

(i.e. algebra homo $D(H) \to \Bbbk$ are in center of $D(H)$)

If $g \otimes \alpha \in G(D(H))$, $g \otimes \alpha$ defines an algebra homomorphism from $D(H) \to \Bbbk$
$\exists\, W \in \mathrm{Rep}(D(H))$ irreducible s.t.

$$\chi_W = g \otimes \alpha$$

$$\chi_{\mathrm{Res}^{D(H)}_H W}(h) = \alpha(h)$$

Then $\mathrm{Res}^{D(H)}_H W \cong \mathbb{1} \iff \alpha = \varepsilon$

- If $W \in \mathrm{Rep}(D(H))$ is 1-dim and $[W : \mathbb{1}]_H \neq 0$, then
$$\chi_W = g \otimes \varepsilon$$
Converse is also true

Lemma: There is 1-1 correspondence between $G(H) \cap Z(H)$ and 1-dimensional rep W of $D(H)$ s.t. $\mathrm{Res}^{D(H)}_H W = \mathbb{1}$

In particular, $W \cong \mathbb{1}_{D(H)} \longleftrightarrow 1_H \in G(H)$

Thm: (Masonka) Let H be a f.d. s.s. Hopf algebra over $\mathbb{C}$ of dim. p^n where $n \in \mathbb{N}$ & p a prime, then H has a nontrivial central group like element
i.e. $G(H) \cap \text{Center}(H) \neq \{1\}$

Proof: (this follows from the correspondence)

By the class equation for s.s. Hopf alg,
$$\dim H = \sum_{W \in \text{Irr}(D(H))} (\dim W) [W : \mathbb{1}]_H$$

But $[\mathbb{1}_{D(H)}, \mathbb{1}]_H = 1$

Then we get
$$p^n = 1 + \sum_{W \neq \mathbb{1}} (\dim W) [W : \mathbb{1}]_H$$

$\dim W \mid \dim H = p^n$

If $\dim W = p^k$, $k > 1$ $\forall$ W, then the equality can't hold.
$\Rightarrow$ $\exists$ 1-dim rep $W \in \text{Rep}(D(H))$
$W \neq \mathbb{1}_{D(H)}$ and $[W : \mathbb{1}]_H \neq 0$

By the lemma, $\exists$ a central grouplike element h, i.e.
$$G(H) \cap \text{Center}(H) \neq \{1\}$$

This allows an inductive way of classifying Hopf algebras of dim $= p^n$.

So, want to classify Hopf of dim p

<u>Thm</u>: [Zhu] Every p-dimensional Hopf alg H over $\mathbb{C}$ is isomorphic to a group algebra, i.e. $H \cong \mathbb{C}[C_p]$

Pf: (1) For 2-dim Hopf algebras, it can be proved directly.

(2) We only consider $p>2$ or p is odd. Suppose H and H^* don't have any nontrivial grouplike element.

Using Radford's formula, $S^4 = Id$
(since no grouplike element)
$\Rightarrow$ $+1, -1$ are the only eigenvalues of S^2.
$\Rightarrow$ trace $(S^2) \neq 0$
(trace (S^2) $\Rightarrow$ $\dim(H)$ is even)

$\Rightarrow$ H is semisimple
$\Rightarrow$ H and H^* have nontrivial grouplike element by last result.

So, we get a contradiction

$\Rightarrow$ H or H^* has a grouplike element.

(3) If $G(H) \neq \{1\}$, $\mathbb{C}[G(H)] = H \cong H^*$

if $G(H^*) \neq \{1\}$, $\mathbb{C}[G(H^*)] = H^* \cong H$

Remark: If $\dim H = p^2$, H can be non-semisimple

eg $H \cong T_{p,\omega}$ $\rightsquigarrow$ Taft algebra of dim p^2

What about the characteristic $\neq 0$ case?
(assuming $\mathbb{k} = \overline{\mathbb{k}}$)

1) $\operatorname{char} \mathbb{k} > p = \dim(H)$, $H \cong \mathbb{k}C_p$
(Etingof - Gelaki)

2) $\operatorname{char} \mathbb{k} = p = \dim H$
$$\begin{cases} H = \mathbb{k}[x]/(x^p) \\ H = \mathbb{k}C_p \\ H = \dfrac{\mathbb{k}[x]}{(x - x^p)} \end{cases}$$
self dual

3) $\operatorname{char} \mathbb{k} < p = \dim H$ (open, believed to be $\mathbb{k}C_p$)

Richard → (hope one day dont have to write this)

§ Cauchy Theorem for $\boxed{\text{s.s.}}$ Hopf algebra and values of indicators

(1) Cauchy's Thm for finite groups
If $p \mid |G|$, there exists $g \in G$ s.t. $\operatorname{ord}(g) = p$, which implies $p \mid \exp(G)$.

Q Does this hold for the exponent of s.s. Hopf algebra?

We know that $\exp(H) \mid (\dim H)^3$ [EG]

$\therefore p \mid \exp(H) \Rightarrow p \mid \dim H$

But, we want

$p \mid \dim H \implies p \mid \exp(H)$

Answer: Yes, due to

(Kashima – Sommerhauser – Zhu)

generalizes to spherical fusion categories

Proof: Let H be a s.s. Hopf algebra.

u = Drinfeld element of $D(H)$

then $u^N = 1$ where $N = \exp(H)$

$= \operatorname{ord}(u)$

$u_W = u_W \cdot Id$, this scalar is a N^{th} root of unity for any irr. rep W of $D(H)$.

scalar

and $\mathbb{Q}(u_W \mid W \in \operatorname{irr}(D(H))) = \mathbb{Q}(\xi_N)$

where $\xi_N = e^{2\pi i/N}$

Let $V \in \operatorname{Rep}(H)$,

$$\nu_n(V) = \underbrace{\frac{1}{\dim H} \sum_{W \in \operatorname{Irr}(D(H))} (\dim W)[W:\mathbb{1}]}_{\in \mathbb{Q}} \underbrace{u_W^n}_{\in \mathbb{Q}(\xi_N)}$$

Let $p \nmid N = \exp(H)$. There exists

$\sigma_p \in \operatorname{Gal}(\mathbb{Q}(\xi_N)/\mathbb{Q})$ s.t.

$\sigma_p : \xi_N \mapsto \xi_N^p$

then $\sigma_p(\nu_n(V))$

$$= \frac{1}{\dim H} \sum_W (\dim W)[W:\mathbb{1}]\, u_W^{np}$$

$$= \nu_{np}(V)$$

$\therefore \quad Gal(\mathbb{Q}(\xi_N)) \curvearrowright \{\nu_n(V) \mid n \in \mathbb{N}\}$

In particular

$$\nu_p(V) = \nu_1(V) = \chi_V(\Lambda) = [V : \mathbb{1}]_H$$

$$\nu_p(H) = [H : \mathbb{1}]_H = 1 \quad \left(\begin{array}{l}\text{since space} \\ \text{of integrals} \\ \text{in } 1\text{-dim}\end{array}\right)$$

Also, $\nu_p(H) = Tr(\alpha)$

where $\alpha : Hom_H(\mathbb{1}, H^{\otimes p}) \to Hom_H(\mathbb{1}, H^{\otimes p})$

$$\sum v_1 \otimes \cdots \otimes v_p \mapsto \sum v_2 \otimes v_3 \otimes \cdots \otimes v_p \otimes v_1$$

easy to see that $\alpha^p = 1$

$Tr(\alpha) = \sum_{i=0}^{p-1} m_i \xi \qquad m_i =$ dimension of eigenspace of eigenvalue ξ_p^i

$$1 = (Tr(\alpha))^p \equiv \sum m_i^p \xi_p^{ip} \mod p$$

$$\equiv \sum_{i=0}^{p-1} m_i^p \underset{\substack{\uparrow \\ \text{Fermat's} \\ \text{little thm}}}{\equiv} \sum_{i=0}^{p-1} m_i$$

$= \dim Hom_H(\mathbb{1}, H^{\otimes p}) \mod p$

$H^{\otimes p} = H \otimes H^{\otimes p-1}$ a left Hopf module

$\Rightarrow \bigoplus \dim(H^{\otimes p-1})$ copies of H

$= \bigoplus (\dim H)^{p-1}$ copies of H.

$\therefore \dim \mathrm{Hom}_H(\mathbb{1}, H^{\otimes p}) = (\dim H)^{p-1} \cdot \dim \mathrm{Hom}_H(\mathbb{1}, H)$

$= (\dim H)^{p-1}$

$\Rightarrow \quad 1 \equiv (\dim H)^{p-1} \mod p$

$\Rightarrow \quad p \nmid \dim H$

$\Rightarrow \Leftarrow$
Contradiction

<u>Remark</u>: Generalized version of Cauchy's thm $\Rightarrow$ Rank finiteness of Modular Tensor Categories

Another consequence is,
If $p \nmid \dim H$, $\nu_p(H) = 1$

<u>Theorem</u>: $\nu_p(H) \neq 1 \iff p \nmid \dim H$

What's the point of this?

Kaplansky conjecture: $\dim V \mid \dim H$

Weaker conjecture: If $p \mid \dim V$, then $p \mid \dim(H)$.

Let H, K s.s over $\mathbb{C}$
$Rep(H)$ and $Rep(K)$ are Morita equivalent if
$$Z(Rep(H)) \underset{\substack{\text{braided}\\ \otimes}}{\cong} Z(Rep(K))$$

$$Rep(D(H)) \underset{\substack{\text{braided}\\ \otimes}}{\cong} Rep(D(K))$$

<u>Thm</u> [N- Schopieray -Wang]
If H and K are s.s. over $\mathbb{C}$ s.t.
$Rep(D(H)) \underset{\substack{\text{braided}\\ \otimes}}{\cong} Rep(D(K))$

then $\nu_n(H) = \nu_n(K)$
for all $n \in \mathbb{N}$.