$(\mathcal{C}, \otimes, a, 1, \rho)$ ⟵ Monoidal category

Ex: $\mathrm{Rep}(G)$

Vec_G^{ω} $\qquad$ ω : 3-cocycle

$\mathrm{Rep}(H)$ $\qquad$ (only class in $H^3(G, \mathbb{k}^{*})$ is important)

$$\underset{\text{tensor functor}}{F} : \mathrm{Vec}_G^{\omega} \longrightarrow \mathrm{Vec}_H^{\omega'}$$

$\mu : G \times G \to \mathbb{k}^{*}$

F gives $\qquad \varphi : G \to H$

$$\frac{\omega}{\varphi^{*}\omega'} = \partial\mu$$

Pointed categories with underlying group G $\longleftrightarrow$ $H^3(G, \mathbb{k}^{*}) / \mathrm{Aut}(G)$

What is this?

can replace by $\mathrm{Out}(G) = \dfrac{\mathrm{Aut}(G)}{\mathrm{Inn}(G)}$

Braiding (commutativity)

$c : X \otimes Y \longrightarrow Y \otimes X$ $\qquad$ an isomorphism of functors

$$\begin{tikzcd}
 & X\otimes(Y\otimes Z) \arrow[r, "c"] & (Y\otimes Z)\otimes X \arrow[dr, "c"] & \\
(X\otimes Y)\otimes Z \arrow[ur, "a"] \arrow[dr, "c"'] & & & Y\otimes(Z\otimes X) \\
 & (Y\otimes X)\otimes Z \arrow[r, "a"'] & Y\otimes(X\otimes Z) \arrow[ur, "c"'] &
\end{tikzcd} \qquad (H1)$$

Hexagon axiom

There is one more hexagon axiom. (H2)
They are independent of each other

Vec_A

has to be abelian in order to have braiding structure

For braiding, we need maps

$$\begin{tikzcd} c : \delta_g \otimes \delta_h \arrow[r, "\sim"] \arrow[d, phantom, "\wr"] & \delta_h \otimes \delta_g \arrow[d, phantom, "\wr"] \\ \delta_{gh} \arrow[r, dashed] & \delta_{hg} \end{tikzcd} \qquad (gh = hg)$$

multiplication by $b(g,h) \in k^*$ (because $\mathrm{Hom}(\delta_{gh}, \delta_{gh}) = k$)

$$H1 \Rightarrow \quad b(g, hk) = b(g,h)\, b(g,k)$$

$$H2 \Rightarrow \quad b(gh, k) = b(g,k)\, b(h,k)$$

Such b are called bicharacters.

Two braided cats with different braidings are "the same"
need notion of braided tensor functor.

$$\begin{tikzcd} F(X \otimes Y) \arrow[r, "F(c)"] \arrow[d, "F_2"'] & F(Y \otimes X) \arrow[d, "F_2"] \\ F(X) \otimes F(Y) \arrow[r, "c"'] & F(Y) \otimes F(X) \end{tikzcd}$$

we want this to commute

Take $F : \mathrm{Vec}(A, b) \xrightarrow{\sim} \mathrm{Vec}(A, b')$

braided $\otimes$ equivalence

$\otimes$ equivalence $\Rightarrow$ $\varphi : A \longrightarrow A$ auto.

to make things easy take $\varphi = \mathrm{Id}_A$.

$\otimes$ $\Rightarrow$ we get $\mu : A \otimes A \longrightarrow \mathbb{R}^*$

since $\mathrm{Vec}(A, b)$ & $\mathrm{Vec}(A, b')$ have same associativity constraint, we want $\partial\mu = 1$

(i.e. its a 2-cocycle)

$$b(g,h)\,\mu(h,g) = \mu(g,h)\,b'(g,h) \qquad *$$

It is possible to find μ s.t. b & b' are related like this.

(need A to have even order)

Plug in $g = h$ in $*$ to get

$$b(g,g)\,\mu(g,g) = \mu(g,g)\,b'(g,g)$$

$$\Rightarrow \quad b(g,g) = b'(g,g)$$

$$\delta_g \otimes \delta_g \xrightarrow{b(g,g)} \delta_g \otimes \delta_g$$

More generality (pointed category)

Vec_A^{ω}

Repeating the same story as above yields notion of abelian 3-cocycles.

$$g \rightsquigarrow \delta_g \otimes \delta_g \xrightarrow{b(g,g)} \delta_g \otimes \delta_g$$

we get a function

$$\rho : A \ni g \longmapsto b(g,g) \in \Bbbk^{\times}$$

Lemma: $A \xrightarrow{\rho} \Bbbk^{\times}$ is a quadratic form.

Defn. of Quadratic form

1. $\dfrac{\rho(xy)}{\rho(x)\rho(y)}$ is bimultiplicative in x and y.
2. $\rho(x^{-1}) = \rho(x)$

Exercise: Prove it

HINT for 1:

$$\delta_x \otimes \delta_y \xrightarrow{c} \delta_y \otimes \delta_x \xrightarrow{c'} \delta_x \otimes \delta_y$$

claim: $cc' = \dfrac{\rho(xy)}{\rho(x)\rho(y)}$

Is there a reconstruction for these?

Theorem: (Joyal - Street)

1. Braided pointed category with underlying group A is determined by (A, ρ) upto braided equivalence.
2. Any quadratic form comes from some braiding.

underlying group $A \rightarrow$ group of isomorphism classes of category is A

pointed $\rightarrow$ objects are invertible

Example: $A = C_2 = \langle \alpha \rangle$

$\rho : C_2 \longrightarrow \mathbb{R}^*$

$\rho(1) = 1$

$\rho(\alpha) = \begin{cases} \pm 1 \\ \pm i \end{cases}$ $\Leftarrow$ gives examples of modular tensor category

(Theorem $\Rightarrow$ quadratic form determines ω, μ)

The braiding c is called symmetric if

$c : X \otimes Y \longrightarrow Y \otimes X \longrightarrow X \otimes Y$ equals $Id_{X \otimes Y}$

Eg: $Rep(G)$ $\quad c(v \otimes w) = w \otimes v$

Summary:
1. monoidal categories
2. braided monoidal categories
3. symmetric monoidal categories

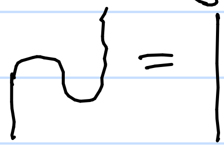
RIGIDITY :

Q How to say categorically that some vector space is finite dimensional ?

Ans : There are 2 maps

$$\mathbb{1} \to V \otimes V^* \qquad\qquad V^* \otimes V \to \mathbb{1}$$

$$1 \mapsto v_i \otimes f_i \qquad\qquad (f, v) \mapsto f(v)$$

 and 

Defn: $\mathcal{C}$ - monoidal category, $X \in \mathcal{C}$

Its (left) dual X^* is an object in $\mathcal{C}$

s.t. $\mathbb{1} \xrightarrow{coev} X \otimes X^*$

$X^* \otimes X \xrightarrow{ev} \mathbb{1}$

s.t. composition

$$\begin{tikzcd} X \arrow[r] \arrow[rrrr, bend right, "Id_X"'] & 1 \otimes X \arrow[r] & X \otimes X^* \otimes X \arrow[r] & X \otimes 1 \arrow[r] & X \end{tikzcd}$$

for some X^*

Its right dual is object ${}^*X$ with maps $1 \to X^* \otimes X$, $X \otimes X^* \to 1$ s.t. two conditions hold.

FACTS: ① X^* if exists is unique up to unique isomorphism.

Defn: $\mathcal{C}$ is called rigid if all $X \in \mathcal{C}$ are right & left rigid.

② If $\mathcal{C}$ is rigid, then $X \to X^*$ is a tensor contravariant functor.

$$(X \otimes Y)^* \cong Y^* \otimes X^*$$

Back to Vec_G^{ω} : This category is rigid with $(\delta_g)^* = \delta_{g^{-1}}$

for V vector space if V f-d. we have $V \xrightarrow{\sim} V^{**}$

- Our current axioms for rigid category don't imply $X \xrightarrow{\sim} X^{**}$

<u>Defn</u>: A pivotal structure on $\mathcal{C}$ is a tensor isomorphism of functors $(X \xrightarrow[\mathrm{Id}]{} X)$ & $(X \xrightarrow[()^{**}]{} X^{**})$

What is it good for?

<u>Ans</u>: With pivotal structure, we can talk about traces.

$$X \xrightarrow{f} X$$

$$\begin{tikzcd}
1 \arrow[r, "\mathrm{coev}_X"] \arrow[rrrrr, bend right, "\mathrm{Tr}(f)"'] & X \otimes X^* \arrow[r, "f \otimes \mathrm{Id}"] & X \otimes X^* \arrow[r] & X^{**} \otimes X^* \arrow[r, "\mathrm{ev}_{X^*}"] & 1
\end{tikzcd}$$

$\mathrm{Tr}(f) \in \mathrm{End}_{\mathcal{C}}(1)$ (Sometimes $\mathrm{End}_{\mathcal{C}}(1) = \mathbb{k}$)

- All usual properties of trace are still true.

- $\dim(X) := \mathrm{Tr}(\mathrm{id}_X)$

Current axioms don't imply $\dim(X) = \dim(X^*)$

<u>Defn:</u> A pivotal structure is spherical if $\dim(X) = \dim(X^*)$ $\forall$ $X \in \mathcal{C}$.