1. TQFT homework from first lecture

Recall that an equivalence between two categories $\mathcal{C}$ and $\mathcal{D}$ is a functor $\mathcal{F} : \mathcal{C} \to \mathcal{D}$ and a functor $\mathcal{G} : \mathcal{D} \to \mathcal{C}$ and natural isomorphisms $\eta_d : \mathcal{F}\mathcal{G}(d) \cong \mathrm{id}_d$ and $\psi_c : \mathcal{G}\mathcal{F}(c) \cong \mathrm{id}_c$.

Also recall that given two categories $\mathcal{C}$ and $\mathcal{D}$ there's a category $\mathrm{Fun}(\mathcal{C}, \mathcal{D})$ whose objects are functors from $\mathcal{C}$ to $\mathcal{D}$ and whose morphisms are natural transformations.

(1) Let G be a finite group, and let BG denote the category with one object $\star$ and $\mathrm{Hom}(\star, \star) = G$ with the group composition. Show that functors $\mathcal{F} : BG \to \mathrm{Vec}$ up to natural isomorphism are in bijection with representations of the group G up to isomorphism.

(2) Construct an equivalence of categories between $\mathrm{Fun}(BG, \mathrm{Vec})$ at the category of representations $\mathrm{Rep}(G)$.

(3) Construct an equivalence between the category of 1-dimensional TQFTs $\mathrm{Fun}(\mathrm{Bord}_1, \mathrm{Vec})$ and the category whose objects are finite dimensional vector spaces and whose morphisms are isomorphisms of vector spaces (not all linear maps!). (Hint: the functor in one direction is given by taking the value on the positively oriented point, and the functor in the other direction we wrote down in lecture.)

(4) Classify unoriented 1-dimensional TQFTs.

(5) TQFTs have values in vector spaces, but we could instead look at functors from the bordism category to other categories. Let Ab be the category of abelian groups. Can you classify all functors $\mathrm{Fun}(\mathrm{Bord}_1, \mathrm{Ab})$?

(6) Can you classify all functors $\mathrm{Fun}(\mathrm{Bord}_1, \mathrm{R{-}mod})$ where R is a commutative ring?

2. TQFT homework from second lecture

(1) Let $\mathcal{F}$ be a 2-dimensional TQFT. Let $\mathcal{G}$ be the 1-dimensional TQFT given by dimensionally reducing a 2-dimensional TQFT along the circle $\mathrm{Bord}_1^{or} \to \mathrm{Bord}_2^{or} \to \mathrm{Vec}$ where the first map is given by crossing with the circle and the second is given by $\mathcal{F}$. Show that $\mathcal{G}$ gives an unoriented TQFT, that is $\mathcal{G}$ can be factored through Bord_1^{unor}.

(2) Show that the two definitions of Frobenius algebra given in the lecture agree.

(3) Let Σ_g be a surface of genus g. Let $Z_{k,a}$ be the TQFT coming from the symmetric Frobenius algebra k with $\tau(1) = a$, and let $Z_{k[x]/x^n}$ be the TQFT coming from the symmetric Frobenius algebra $k[x]/x^n$ with the trace being the coefficient of x^{n-1}. Compute $Z_{k,a}(\Sigma_g)$ and $Z_{k[x]/x^n}(\Sigma_g)$.